Entangled Simultaneous Measurement and Elementary Particle Representations

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Abstract

It is proposed that the principles of relativistic quantum mechanics are incomplete for simultaneous measurement of non-commuting operators. Consistent joint measurement of incompatible observables at a single point in space-time requires that the system be in an entangled state with vacuum meters. Entangled simultaneous measurement for non-commuting observables is suggested as the basis for observed fermionic multiplets. This generalizes the standard spin representations for particles arising from Lorentz invariance. It is shown that operator entanglement for all quantum observables in the Poincare algebra, coupled with Fermi-Dirac statistics, mandates six fermions. The quark and lepton generations are proposed to form a super-structure of the Poincare algebra based on the principle of entangled simultaneity. Mathematically, the structure is known as a Naimark extension. The required entanglement between particle generations for left-handed quarks is observed in the Cabibbo-Kobayashi-Maskawa matrix. In the appendix, the Naimark-extended von Neumann lattice is shown to be distributive, thereby suggesting the principle of entangled simultaneity as a mechanism to avoid quantum non-locality. Keywords: entangled simultaneous quantum measurement, Naimark extension, lepton/quark generations. PACS Number: 03.65.BZ. Please direct correspondence to R.Y. Levine, bob@spectral.com.

1 Introduction

The measurement of non-commuting operators in a quantum system is at the root of the well-known paradoxes of quantum mechanics. For example, the Kochen-Specker paradox involves measurement of a set of non-commuting operators with mutually commuting subsets [1, 2]. The Bell inequalities are similarly formulated by incompatible space-like separated spin measurements [2, 3]. The connection of these problems to quantum non-locality motivates the consideration of an alternative definition for the measurement of incompatible quantum observables. Because measured quantities are inherently statistical, quantum mechanics requires an ensemble of identical systems to establish an expectation value. The types of measurements on the ensemble in the situation of non-commuting observables is critical to the interpretation of the result. For example, the usual simultaneous measurement of position and momentum would require separate measurements of each operator on half the ensemble at the same time. This procedure for the assignment of position and momentum to a system, coupled with wavefunction collapse, leads to a built-in non-locality in system observables. One approach to this problem is to restrict *valid* measurement of non-commuting observables to be with special-purpose ancillary quantum systems (referred to as vacuum meters in this paper) that are entangled with the original system. In the alternative *entangled simultaneous* measurement scheme, all systems in the ensemble have the same experimental setup with coupling to vacuum meters for a joint position/momentum measurement. The premise of this paper is that this procedure, while avoiding quantum non-locality, yields results that are properly interpreted as a joint measurement of non-commuting observables. Furthermore, it is assumed that, probably due to a principle involving quantum non-locality, entangled states are fundamental to particle representations. The principle mandates that particle states allow the simultaneous determination of all particle observables. These assumptions impose the structure of a Naimark extension on particle multiplets.

Allowed quantum numbers and statistics for relativistic particles are strictly constrained by the dual principles of Lorentz invariance and quantum mechanics [4]. The latter requires states that repre-

sent symmetries of the lagrangian. A larger structure, the Naimark extension of the Poincare algebra [5, 6], results if states are required for the measurement of incompatible observables with entangled vacuum meters. New commuting operators are defined in a Naimark extension that project to the original set upon meter measurement. A key component of the measurement scheme is the entanglement of the system with a vacuum state containing independent meters. The assumption of Fermi-Dirac statistics for the meters forces a different particle identity (flavor) onto the ancillary particles that make up the meters. It is shown in this paper that the minimum Hilbert space for the Naimark extension of the Poincare algebra contains six independent fermions, which are identified with left- and right-handed lepton/quark generations. The particle set is the minimum required for realization of the Poincare algebra on commuting operators such that particles can entangle with vacuum meters. The universal nature of fermionic multiplets, existing for both leptons and quarks, motivates the suggestion of a single underlying principle rooted in relativistic quantum mechanics. An elementary particle is interpreted as a complex, entangled quantum system in which the entire space of Poincare observables is realized on commuting operators.

The Naimark extension or embedding is a mathematical description of measurement with a quantum apparatus [7]. In the original construction non-orthogonal projection operators, such as generated by optical coherent states [8], are extended to orthogonal projection operators in a combined system/meter Hilbert space [9]. As first proposed by von Neumann [7], and developed by Arthurs and Kelly [10], a realization of the Naimark extension for position and momentum is obtained by the entangling of a harmonic oscillator with meter harmonic oscillators in ground states. In the Arthurs-Kelly model, measurement on two meters results in the collapse of the system wavefunction to a coherent state corresponding to the measured position and momentum. As explained by Levine and Tucci [11], entangled simultaneous measurement of position and momentum with a single meter results in the collapse of the system to the eigenstate of the operator that was measured on the system. However, the system/meter expectation values are still proportional to

the desired system expectation values. The theory of entangled simultaneous quantum measurement was extended to non-relativistic spin by coupling to spin-1/2 meters by Levine and Tucci [12]. In this case measurements project the system to Bloch states corresponding to the measured spin components. Analogous simultaneous spin measurement schemes are found in Refs. [13]-[16]. Modified Stern-Gerlach experiments, based on hamiltonian models for simultaneous spin measurement, are discussed in Ref. [17]. The relativistic generalization of position, momentum, and angular momentum measurements leads to the consideration of the Poincare algebra of observables and elementary particles. In an attempt to describe elementary particle multiplets, Levine [18] suggested the generalization of spin ($SU(2)$) measurement to the measurement of $SO(2n)$ Clifford algebra operators using $2n$ spin meters. A more fundamental application to elementary particle multiplets, originating in the structure of the Poincare algebra, is proposed in this paper. Finally, a general discussion of entangled simultaneous measurement with second quantized relativistic fields is found in Ref. [19].

The mechanism for entangled simultaneous measurement is particularly transparent for second quantized systems. The vacuum, which is defined as the state projected to zero by annihilation operators, is critical to isolate incompatible system observables on independent meters. Another key property of the Naimark extension is that the minimal extension is determined from the pattern of commutativity in the algebra of observables. These properties are demonstrated for non-relativistic position, momentum, and angular momentum measurements in Section 2. In the latter case the three components of angular momentum are measured through the entanglement of the system with two vacuum meters. The formalism for first quantized harmonic oscillators, reviewed in the section, is directly generalized to relativistic fields. Section 3 contains the relativistic generalization of simultaneous measurement for Dirac fermions. It is shown that the Naimark extension of the Poincare algebra consists of six independent fermionic fields. The structure results from the embedding of the three 3-vector operators of momentum, angular momentum, and boost generators. It is suggested that the three entangled lepton/quark generations, an entanglement

that is described for left-handed quarks by the Cabibbo-Kobayashi-Maskawa matrix [20], is a result of the Naimark extension of the Poincare algebra for massive fermions. Appendices A and B contain the formal details of Naimark extensions for second quantized angular momentum and Dirac fields, respectively.

As mentioned above, non-commuting operators are the basis for observed non-locality in quantum systems. The quantum logical consequence of incompatible operators is the non-distributivity of the von Neumann subspace lattice of quantum measurement outcomes [7, 21]. The possibility of a deeper reality, involving a new principle of quantum measurement, motivates the suggestion that measurement of non-commuting observables must be confined to Naimark-extended Hilbert spaces. In Appendix C, it is shown that this principle of entangled simultaneity can be applied to the subspace lattice to avoid paradoxical (non-distributive) statements. As an example, the distributive Naimark-extended lattice is constructed for spin-1/2 measurement. A conclusion follows in Section 4.

2 Non-relativistic Systems

In this section the simultaneous measurement of position and momentum in a first quantized harmonic oscillator, and of angular momentum in a second quantized system, are considered. Both cases involve the construction of non-relativistic Naimark extensions, and have properties that generalize to the relativistic case.

An example of a non-relativistic Naimark extension without second quantization uses a pair of one-dimensional harmonic oscillators [12] with the hamiltonian ($\hbar = c = 1$)

$$H = \sum_{j=1}^{2} \left(\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 q_j^2 \right). \tag{1}$$

In terms of annihilation operators, $a_j = \sqrt{m_j \omega_j / 2}(q_j + i p_j / m_j \omega_j)$, the expression in Eq. (1) is written,

$$H = \sum_j \omega_j a_j^\dagger a_j = \vec{a}^\dagger D \vec{a}, \tag{2}$$

where $\vec{a}^\dagger = (a_1^\dagger, a_2^\dagger)$,$D = diag(\omega_1, \omega_2)$, and the constant zero point energy is dropped. A rotation by angle θ, $\vec{A} = R(\theta) \cdot \vec{a}$, is applied to these operators to define another Hilbert space with combined hamiltonian, $H = H_0 + H_{int}$, given by

$$H_0 = (c^2\omega_1 + s^2\omega_2)A_1^\dagger A_1 + (s^2\omega_1 + c^2\omega_2)A_2^\dagger A_2, \tag{3}$$

and

$$H_{int} = sc(\omega_1 - \omega_2)(A_1^\dagger A_2 + A_2^\dagger A_1), \tag{4}$$

with $s = \sin\theta$ and $c = \cos\theta$. Note that, from the commutativity of a_1 and a_2, the operators A_1 and A_2 commute. Through this simple rotation, the operators $a_j(t) = e^{-i\omega_j t}a_j(0)$, $j = 1, 2$, combine incompatible information about the A_1 system into operators that are simultaneously measured. For example, consider the commuting operators $q_1(t) = (a_1(t) + a_1^\dagger(t))/\sqrt{2m_1\omega_1}$, and $p_2(t) = -i\sqrt{m_2\omega_2/2}(a_2(t) - a_2^\dagger(t))$ for system 1 position and system 2 momentum, respectively. In terms of the operators $A_1(0)$ and $A_2(0)$ at time zero, these operators are given by

$$q_1(t) = \frac{1}{\sqrt{2m_1\omega_1}}(e^{-i\omega_1 t}(cA_1(0) + sA_2(0)) + e^{i\omega_1 t}(cA_1(0) + sA_2(0))^\dagger), \tag{5}$$

and

$$p_2(t) = i\sqrt{m_2\omega_2/2}(e^{-i\omega_2 t}(sA_1(0) - cA_2(0)) - e^{i\omega_2 t}(sA_1(0) - cA_2(0))^\dagger). \tag{6}$$

The expectation of the operators $q_1(t)$ and $p_2(t)$ in the state $|\psi>_{A_1} |0>_{A_2}$, in which the A_2 system is in the vacuum, is given by

$$\begin{aligned} < q_1(t) > &= \frac{c}{\sqrt{2m_1\omega_1}}[\cos(\omega_1 t) < \psi|(A_1(0) + A_1^\dagger(0))|\psi > - \\ &\quad i\sin(\omega_1 t) < \psi|(A_1(0) - A_1^\dagger(0))|\psi >] \end{aligned} \tag{7}$$

and

$$\begin{aligned} < p_2(t) > &= -is\sqrt{\frac{m_2\omega_2}{2}}[i\sin(\omega_2 t) < \psi|(A_1(0) + A_1^\dagger(0))|\psi > - \\ &\quad \cos(\omega_2 t) < \psi|(A_1(0) - A_1^\dagger(0))|\psi >]. \end{aligned} \tag{8}$$

Inversion of the expressions in Eqs.(7) and (8) for the measurement of $(A_1(0)+A_1^\dagger(0))$ and $(A_1(0)-A_1^\dagger(0))$ on commuting operators $q_1(t)$ and $p_2(t)$ is given by

$$\begin{bmatrix} < A_1(0) + A_1^\dagger(0) > \\ < A_1(0) - A_1^\dagger(0) > \end{bmatrix} = \frac{-1}{\cos((\omega_1 - \omega_2)t)} \begin{vmatrix} -\cos\omega_2 t & i\sin\omega_1 t \\ -i\sin\omega_2 t & \cos\omega_1 t \end{vmatrix} \times \begin{bmatrix} (\sqrt{2m_1\omega_1}/c) < q_1(t) > \\ (i\sqrt{2}/(\sqrt{m_2\omega_2}s)) < p_2(t) > \end{bmatrix}. \tag{9}$$

The rotation $\vec{A} = R(\theta)\cdot\vec{a}$ corresponds to the system/meter entanglement that defines a Naimark extension [8]. In this construction, a meter Hilbert space $\mathcal{H_M}$ is combined with the system space $\mathcal{H_S}$ to define the space $\mathcal{H_M}\otimes\mathcal{H_S}$ in which relevant operators commute. For the case of position and momentum, commuting operators can be defined as $Q_T = Q_1 - Q_2$ and $P_T = P_1 + P_2$, where the subscripts 1 and 2 correspond to system and meter, respectively. Information of the system position and momentum is contained in the orthonormal eigenstates $\{|\xi,\eta>\}$ of (Q_T, P_T) in $\mathcal{H_M}\otimes\mathcal{H_S}$. Mathematically, the projection property of a Naimark extension is defined in terms of density operators $\rho_{S+M} = |\xi,\eta><\xi,\eta|$ and $\rho_M = |0><0|$, which correspond to Naimark and meter states, respectively. Projection to a system coherent state [22] is then given by

$$|\alpha><\alpha| = Trace_M(\rho_{S+M}\rho_M), \tag{10}$$

where $\alpha = \sqrt{m\omega/2}(q + ip/(m\omega))$ for a system of mass m, position q, and momentum p; and where $Trace_M$ is the trace over meter states. The projection property in Eq.(10), a defining characteristic of Naimark extensions, is the basis for entangled simultaneous measurement of position and momentum. In a harmonic oscillator model of the measurement, Arthurs and Kelly [10] observed that the entanglement of the system with two vacuum meters resulted in the collapse of the system to a coherent state upon position and momentum measurement on meters. However, the single system/meter entanglement above, $\vec{A} = R(\theta)\cdot\vec{a}$, is sufficient to simultaneously measure position and momentum. Of course in this case the system is projected onto an eigenstate of the measured operator rather than a coherent state. This realization of the Naimark extension, which

is the minimum possible for entangled simultaneous measurement, is used in this paper.

In Appendix A the above construction is extended to a second quantized system described by an angular momentum basis set $\{|jm>; j = 0, 1, \ldots; m = -j, \ldots, j\}$. It is shown that, by entangling the system to two ancillary meters in vacuum states $|0>$, where the vacuum is defined with zero occupation of the states $\{|jm>\}$, the original system angular momentum components are simultaneously measured. Extended operators $\tilde{J}_x^{(1)}$, $\tilde{J}_y^{(2)}$, and $\tilde{J}_z^{(3)}$, corresponding to the original system (1) and meters (2) and (3), have a role similar to Q_T and P_T defined above. These operators commute and project onto the system operators J_x, J_y, and J_z upon meter measurement. An analogy to the Naimark projection property in Eq. (10) is also described.

The examples in this section and Appendix A demonstrate the difference between the formulation of entangled simultaneous measurement in first and second quantized systems. In the former case, which has found applications in quantum optics [23]-[25], the entangling hamiltonian in Eqs. (3) and (4) is dependent on the measured position and momentum operators. In a second quantized system the entangling interaction is independent of the measured operators, which are represented as bilinear functions of creation and annihilation operators (see Eqs.(27)-(29)). Because of this property, the structure of the Naimark extension depends only on the *algebra* represented by the observables. The minimal extension requires a sufficient number of independent meters for the measurement of non-commuting operator sets. For example, the Naimark extension of the combined galilean algebra of position Q_i, momentum P_i, and angular momentum J_i, $i = x, y, z$, appropriate for a non-relativistic particle [4], is derived from the commutation relations $j, k, m = x, y, z$,

$$\begin{array}{lll} [Q_k, P_j] = i\delta_{kj} & [J_k, Q_j] = i\epsilon_{kjm} Q_m & [J_k, P_j] = i\epsilon_{kjm} P_m \\ [J_k, J_j] = i\epsilon_{kjm} J_m & [Q_k, Q_j] = 0 & [P_k, P_j] = 0. \end{array} \tag{11}$$

A second quantized Naimark extension for the above algebra consists of three distinguishable particles upon which mutually commuting operator pairs (J_i, P_i),$i = x, y, z$, are measured. All three commuting components of the position vector Q_i,$i = x, y, z$, are

measured on a fourth particle. The four entangled particles of the minimum Naimark extension result from the mutually commuting subsets of the algebra in Eq.(11). This construction is generalized to the Poincare algebra in the next section.

3 Relativistic Systems

The previous section contains constructions for entangled simultaneous measurement of position, momentum, and angular momentum with quantum meters. The relativistic generalization of these results is the entangled measurement of operators in the Poincare algebra, from which particle representations satisfying Lorentz invariance are obtained [4]. It is suggested that the Naimark extension of this algebra is the basis for a larger structure than the usual particle spin representations, and is the source of the quark/lepton multiplets. Appendix B contains the formulation of entangled simultaneity for fermions described by second quantized Dirac fields. It is shown that n non-commuting operators $\{\Theta_k, k = 1, \ldots, n\}$ are simultaneously measurable by the system entanglement with $(n-1)$ fermions. The Naimark-extended state for this measurement is $|\phi>_{(1)}\ |0>_{(2)}\ \ldots |0>_{(n)}$, in which the ancillary particles $(j), j = 2, \ldots, n$ are in the vacuum state.

The relativistic generalization of position and momentum is the Poincare algebra of operators involving the generators of Lorentz transformations. The nine operators, not including the hamiltonian, can be grouped into three 3-vectors; momentum,

$$P_j = i \int d\vec{x} \psi^\dagger(\vec{x}, t) \partial_j \psi(\vec{x}, j), \tag{12}$$

angular momentum, $J_k = i\epsilon_{kjm} J^{jm}$, $k, j, m = 1, 2, 3$, with

$$J^{jm} = i \int d\vec{x} \psi^\dagger(\vec{x}, t)(x^j \partial^m - x^m \partial^j - i\mathcal{J}^{jm})\psi(\vec{x}, t), \tag{13}$$

and boost generators,

$$K_j = \int d\vec{x} \psi^\dagger(\vec{x}, t)(it\partial_j + ix_j \partial_0 + \mathcal{K}_j)\psi(\vec{x}, t), \tag{14}$$

with (assuming Pauli matrices σ_k, $k = 1, 2, 3$)

$$\mathcal{J}^{jm} = i\epsilon^{jmk} \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix}, \tag{15}$$

and

$$\mathcal{K}_j = i/2 \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix}. \tag{16}$$

The boost operators in Eq.(14) contain the average position of particle energy through the integral $\int d\vec{x}\vec{x}\psi^\dagger H\psi$. The commutation relation $[P_k, K_j] = i\delta_{jk}H$ further suggests the identification of $\vec{K}$ with position. However the boost operators are not hermitian. Consequently, in order to describe the Poincare algebra with measurable observables, the real and imaginary parts are separately defined as $K_j = K_j^1 + iK_j^2$ with K_j^i, $i = 1, 2$, hermitian. The operator K_j^1 is the physical location of the particle energy, and K_j^2 reflects spin state effects in the boost operator.

The operators in Eqs.(12)-(15) fit the model in Eq.(39) for which a second quantized Naimark extension is discussed in Appendix B. The complete Poincare algebra in terms of boost operators K_j^i, $i = 1, 2$, is given by

$$\begin{aligned}
&[J_j, J_k] = i\epsilon_{jkm}J_m, && [J_j, K_k^1] = i\epsilon_{jkm}K_m^1, && \left[K_i^2, P_j\right] = 0, \\
&[J_j, P_k] = i\epsilon_{jkm}P_m, && [J_j, K_k^2] = i\epsilon_{jkm}K_m^2, && \left[K_i^2, H\right] = 0, \\
&\left[K_j^1, H\right] = iP_j, && [J_i, H] = 0, && [P_i, H] = 0, \\
&\left[K_j^1, P_k\right] = i\delta_{jk}H, && \left[K_j^1, K_k^2\right] + \left[K_j^2, K_k^1\right] = 0, && \\
& && \left[K_j^1, K_k^1\right] - \left[K_j^2, K_k^2\right] = -i\epsilon_{jkm}J_m. &&
\end{aligned} \tag{17}$$

A minimal embedding of the Poincare algebra for massive fermions uses the commutations $[J_j, K_j^1] = [P_j, K_j^2] = 0$,$j = 1, 2, 3$, in Eq. (17). Three independent fields ψ_j are sufficient for (J_j, K_j^1), $j = 1, 2, 3$, measurements; and on an additional three fields ψ'_j the operators (P_j, K_j^2), $j = 1, 2, 3$, are measured. The hamiltonian operator, satisfying $[H, P_j] = [H, K_j^2] = 0$, could be measured on any of the ψ'_j fields. Because of the time derivative in the boost operator in Eq. (14), the energy density position corresponds to the original

mass (before entanglement, m_j in Eq.(34)) rather than the system mass (M_1 in Eq.(35)). This suggests that the three fields ψ_j used to measure boost operators should have the same mass as the original system, which implies constraints on the entanglement in Eqs. (34)-(36).

The Naimark embedding of the Poincare algebra suggests an underlying explanation for the generation structure of quarks and leptons. The fermions (u, c, t), (d, s, b), (e, μ, τ), and $(\nu_e, \nu_\mu, \nu_\tau)$ are triplets within which particles differ only by mass. A realization of the Naimark extension is obtained by identifying massive quarks and leptons with the fields $\psi_j(\psi'_j)$, $j = 1, 2, 3$, to obtain the mapping

$$\begin{bmatrix} f_L^1 & f_L^2 & f_L^3 \\ f_R^1 & f_R^2 & f_R^3 \end{bmatrix} \longrightarrow \begin{bmatrix} (J_1, K_1^1) & (J_2, K_2^1) & (J_3, K_3^1) \\ (P_1, K_1^2) & (P_2, K_2^2) & (P_3, K_3^2) \end{bmatrix} \tag{18}$$

where $f^i = (u, c, t), (d, s, b)$, and (e, μ, τ). The entanglement between different generations of left and right-handed fermions is the result of spontaneous symmetry breaking of the vacuum by Higgs particles. As described in Ref. [26], the entanglement for quarks results from the diagonalization of the coupling to Higgs particles expressed as rotations $\vec{f}'_R = W_{u(d)} \cdot \vec{f}_R$ and $\vec{f}'_L = U_{u(d)} \cdot \vec{f}_L$, where $u(d)$ corresponds to *up (down)* quarks, and $\vec{f} = (f^1, f^2, f^3)^T$ corresponds to the triplets (u, c, t) and (d, s, b). The Cabibbo-Kobayashi-Maskawa matrix, $V = U_u^\dagger U_d$, is the only observable (other than mass) arising from fermionic mixing in the standard model [20]. Mass generation from a non-zero Higgs vacuum expectation provides the entanglement connecting left- and right-handed particles that completes the fermionic Naimark extension.

Massless fermions form representations of a reduced Poincare algebra [4]. For left-handed particles (like massless neutrinos) with fixed $+1/2$ helicity, the only measurements required in an inertial frame are boosts K_j^1 and momentum P_j operators. From the commutation $[P_j, K_i^1] = 0$ for $i \neq j$, the operator correspondence of left-handed fermions

$$(\nu_e, \nu_\mu, \nu_\tau) \longrightarrow ((P_1, K_2^1), (P_2, K_3^1), (P_3, K_1^1)), \tag{19}$$

is a sufficient Naimark extension.

4 Conclusions

In this paper it is suggested that quantum mechanics is incomplete, and the complete formulation is relevant at relativistic energies. A complete quantum system is in an entangled state with the property that the entire observable phase space of non-commuting operators is measurable.

The definition of a relativistic fermion as a minimally entangled system for the representation of the Poincare algebra is examined. It is shown that the generation structure of leptons and quarks fits this definition with entanglement provided by Higgs couplings as observed in the Cabibbo-Kobayashi-Maskawa matrix. In the high energy limit, the primary structure is not a particle, but rather a system of six entangled particles upon which the complete phase space is represented. This is a generalization of the usual group theoretical particle representation arising from Lorentz invariance of the lagrangian.

A Entanglement for Angular Momentum

In this appendix we construct the Naimark extension for angular momentum measurement in a second quantized system described by angular momentum states $\{|jm>; j = 0, \ldots; m = -l, \ldots, l\}$ with a non-interacting hamiltonian given by

$$H = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \sum_{k=1}^{3} E_k(j) a_{jm}^{(k)\dagger} a_{jm}^{(k)}, \tag{20}$$

where $a_{jm}^{(k)}$, $k = 1, 2, 3$, are the annihilation operators for three independent systems satisfying commutation relations

$$\left[a_{jm}^{(k)}, a_{j'm'}^{(k')\dagger}\right] = \delta_{jj'}\delta_{mm'}\delta_{kk'}, \quad \left[a_{jm}^{(k)}, a_{j'm'}^{(k')}\right] = 0. \tag{21}$$

The system/meter operators are defined by a rotation of the original system as

$$\begin{bmatrix} A_{jm}^{(1)} \\ A_{jm}^{(2)} \\ A_{jm}^{(3)} \end{bmatrix} = R \cdot \begin{bmatrix} a_{jm}^{(1)} \\ a_{jm}^{(2)} \\ a_{jm}^{(3)} \end{bmatrix}, \tag{22}$$

where R is a $|jm>$-independent rotation matrix. Substitution of Eq. (22) into Eq. (20), with $E = diag(E_1(j), E_2(j), E_3(j))$, yields a system/meter hamiltonian given by

$$H = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \sum_{k=1}^{3} \sum_{k'=1}^{3} A_{jm}^{(k)\dagger} D_{kk'}(j) A_{jm}^{(k')}, \tag{23}$$

where

$$D(j) = RE(j)R^{\dagger}. \tag{24}$$

The expression in Eq. (23) can be written as a non-interacting hamiltonian (terms with $k = k'$),

$$H_0 = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \sum_{k=1}^{3} A_{jm}^{(k)\dagger} D_{kk}(j) A_{jm}^{(k)}, \tag{25}$$

and an interaction term

$$\begin{aligned} H_{int} &= \sum_{j=0}^{\infty} \sum_{m=-j}^{j} [D_{12}(j) A_{jm}^{(1)\dagger} A_{jm}^{(2)} + D_{13}(j) A_{jm}^{(1)\dagger} A_{jm}^{(3)} \\ &+ D_{23}(j) A_{jm}^{(2)\dagger} A_{jm}^{(3)}] + h.c. \end{aligned} \tag{26}$$

Assume that the operators $A_{jm}^{(1)}$ and $A_{jm}^{(k)}$, $k = 2, 3$, correspond to the system and meters, respectively, and consider the commuting angular momentum operators for the *original* independent systems in Eq.(20),

$$\tilde{J}_x^{(1)} = \sum_j \sum_{mm'} a_{jm}^{(1)\dagger}(t) a_{jm'}^{(1)}(t) < jm|\mathcal{J}_x|jm' >, \tag{27}$$

$$\tilde{J}_y^{(2)} = \sum_j \sum_{mm'} a_{jm}^{(2)\dagger}(t) a_{jm'}^{(2)}(t) < jm|\mathcal{J}_y|jm' >, \tag{28}$$

and

$$\tilde{J}_z^{(3)} = \sum_j \sum_{mm'} a_{jm}^{(3)\dagger}(t) a_{jm'}^{(3)}(t) < jm|\mathcal{J}_z|jm' >, \tag{29}$$

where $\mathcal{J}_\alpha$, $\alpha = x, y, z$, are matrix representations of the angular momentum components. The *tilde* notation in Eqs. (27)-(29) is used to emphasize the distinction between the operators in the entangled space and the system. The time dependence of $a_{jm}^{(k)}$,

given by $a^{(k)}_{jm}(t) = e^{-iE_k(j)t}a^{(k)}_{jm}(0)$, indicates that the operators in Eqs. (27)-(29) are time independent. The substitution of Eq. (22) into Eqs. (27)-(29), and evaluation in the state $|\psi>_{(1)}\ |0>_{(2)}\ |0>_{(3)}$ in which the meters (2) and (3) are in vacuum states and the system (1) is in the state $|\psi>$, results in the expectation values,

$$\begin{aligned} <\tilde{J}^{(1)}_x> &= \sum_j \sum_{mm'} <\psi|A^{(1)\dagger}_{jm}(t)A^{(1)}_{jm'}(t)|\psi><jm|\mathcal{J}_x|jm'>|R_{11}|^2 \\ &= |R_{11}|^2 <\psi|J^{(1)}_x|\psi>, \end{aligned} \tag{30}$$

$$\begin{aligned} <\tilde{J}^{(2)}_y> &= \sum_j \sum_{mm'} <\psi|A^{(1)\dagger}_{jm}(t)A^{(1)}_{jm'}(t)|\psi><jm|\mathcal{J}_y|jm'>|R_{12}|^2 \\ &= |R_{12}|^2 <\psi|J^{(1)}_y|\psi>, \end{aligned} \tag{31}$$

and

$$\begin{aligned} <\tilde{J}^{(3)}_z> &= \sum_j \sum_{mm'} <\psi|A^{(1)\dagger}_{jm}(t)A^{(1)}_{jm'}(t)|\psi><jm|\mathcal{J}_z|jm'>|R_{13}|^2 \\ &= |R_{13}|^2 <\psi|J^{(1)}_z|\psi>\ . \end{aligned} \tag{32}$$

The expressions in Eqs. (30)-(32) demonstrate the simultaneous measurement of system angular momentum components on the commuting operators $\tilde{J}^{(1)}_x$, $\tilde{J}^{(2)}_y$, and $\tilde{J}^{(3)}_z$. The Naimark projection property corresponding to Eq. (10) is given by

$$J^{(1)}_i = Trace_{(2)(3)}\left[\frac{\tilde{J}^{(i)}_i \rho_M}{|R_{1i}|^2}\right], i = 1, 2, 3, \tag{33}$$

where $J^{(i)}_i$ is the i^{th} component of angular momentum for the system, $\rho_M = |0><0|$ is the density matrix for the product vacuum state $|0> = |0>_{(2)}\ |0>_{(3)}$, and the trace is over the Hilbert space for meters (2) and (3).

B Entanglement for Dirac Fields

Consider the Dirac hamiltonian for n independent fermionic fields,

$$H = \sum_{j=1}^{n} \int d\vec{x}(\pi_j\gamma^0\vec{\gamma}\cdot\vec{\nabla}\psi_j + m_j\pi_j\gamma^0\psi_j), \tag{34}$$

where $\psi_j(\vec{x},t)$, $j=1,\ldots,n$ are second quantized Dirac spinors, γ^μ are Dirac matrices (Ref. [4] notation), and $\pi_j = i\psi_j^\dagger$ is the conjugate momentum to the field ψ_j. The rotation of the vector of fermionic fields, $\vec{\psi}^\dagger = (\psi_1^\dagger,\ldots,\psi_n^\dagger)$, given by $\vec{\Psi} = R\cdot\vec{\psi}$, defines the system/meter Hilbert spaces. Substitution into Eq. (34) results in a hamiltonian $H = H_0 + H_{int}$ with

$$H_0 = \sum_{j=1}^{n} \int d\vec{x}(\Pi_j\gamma^0\vec{\gamma}\cdot\vec{\nabla}\Psi_j + M_j\Pi_j\gamma^0\Psi_j), \tag{35}$$

and

$$H_{int} = \int d\vec{x}(\vec{\Pi}\gamma^0(\mathcal{M} - diag(M_1,\ldots,M_n))\vec{\Psi}), \tag{36}$$

with $\Pi_j = i\Psi_j^\dagger$, $m = diag(m_1,\ldots,m_n)$, $\mathcal{M} = RmR^\dagger$, and $M_j = \mathcal{M}_{jj}$.

The free fields ψ_j, $j=1,\ldots,n$, satisfy the equation,

$$\psi_j(\vec{x},t) = e^{iH_0(j)t}\psi_j(\vec{x},0)e^{-iH_0(j)t}, \tag{37}$$

where $H_0(j)$ is the j^{th} term in Eq. (34). The second quantized field in Eq. (37) is written in terms of the vector $\vec{\Psi}(\vec{x},0)$, at time zero, as

$$\psi_j(\vec{x},t) = e^{iH_0(j)t}(R^\dagger\vec{\Psi}(\vec{x},0))_j e^{-iH_0(j)t}. \tag{38}$$

Consider the entangled simultaneous measurement of field operators,

$$\Theta_k = \int d\vec{x}\psi_1^\dagger(\vec{x},t)\theta_k(\vec{x})\psi_1(\vec{x},t), k = 1,\ldots,n, \tag{39}$$

where $\theta_k(\vec{x})$ are 4×4 $\vec{x}$-dependent operators. The substitution of Eq. (38) into Eq. (39), with ψ_1 replaced with ψ_k, results in the expression,

$$\tilde{\Theta}_k^{(k)}(t) = e^{iH_0(k)t}\int d\vec{x}(\vec{\Psi}^\dagger(\vec{x},0)R^\dagger)_k\theta_k(\vec{x})(R\vec{\Psi}(\vec{x},0))_k e^{-iH_0(k)t}. \tag{40}$$

Note that the operators $\tilde{\Theta}_k^{(k)}$,$k = 1,\ldots,n$, are mutually commuting from the independence of the fields ψ_k. The commuting set of operators, $\tilde{\Theta}_k^{(k)}(0) = e^{-iH_0(k)t}\tilde{\Theta}_k^{(k)}(t)e^{iH_0(k)t}$, evaluated in the state $|\phi>_{(1)}|0>_{(2)}\ldots|0>_{(n)}$, where (k) corresponds to the system with field Ψ_k, results in the expression,

$$<\tilde{\Theta}_k^{(k)}(0)> = <\phi|\int \Psi_1^\dagger(\vec{x},0)\theta_k(\vec{x})\Psi_1(\vec{x},0)|\phi>|R_{1k}|^2. \tag{41}$$

The expression in Eq. (41) represents the entangled simultaneous measurement of Θ_k, $k = 1, \ldots, n$, in the state $|\phi >_{(1)}$ through the Naimark extension to n fermions. The equation is a relativistic generalization to an arbitrary set of operators of Eqs. (30)-(32) for non-relativistic angular momentum. The Naimark projection property in Eqs. (10) and (33) is generalized to the relativistic case by the condition

$$\Theta_k = Trace_{(2)(3)\ldots(n)} \left[\frac{\tilde{\Theta}_k^{(k)}(0)\rho_M}{|R_{1k}|^2} \right], k = 1, \ldots, n, \tag{42}$$

where $\rho_M = |0 >< 0|$ for the meter state $|0 >= |0 >_{(2)} \ldots |0 >_{(n)}$.

C Quantum Logical Implications

Measurement of non-commuting operators in a quantum system is the cause of the non-distributivity in the von Neumann subspace lattice. One approach to the inconsistencies and paradoxes arising from this property is to restrict the interpretation of these measurements as simultaneous only if joint measurements occur on an entangled system at a fixed space-time location. In this appendix, this procedure for the measurement of incompatible observables is included in the subspace lattice of logical propositions. It is shown that the Naimark-extended von Neumann lattice is distributive for a simple example of spin-1/2 component measurement.

C.1 Entangled Simultaneous Spin Measurement

In this section the entangled simultaneous measurement of spin-1/2 operators S_x and S_z is demonstrated by the coupling to an independent spin-1/2 meter. This derivation is a simplified version of the mechanism proposed by Levine and Tucci [12].

Consider two independent spin-1/2 systems S and R defined by commuting spin-1/2 operators S_j and R_j, $j = 1, 2, 3$, which are both expressed as Pauli matrices $\sigma_j/2$, ($\hbar = 1$). Assume an entangling hamiltonian given by

$$H = kS_xR_z, \tag{43}$$

where k is an arbitrary coupling constant. The evolution operator $U = e^{-iHt}$, corresponding to the hamiltonian H in Eq. (43), is given by

$$U = (c - 4isS_xR_z) \tag{44}$$

where $c = \cos(kt/4)$ and $s = \sin(kt/4)$. The evolution of operators $S_z(t)$ and $R_x(t)$ in the Heisenberg picture determines the entangled spin component measurement at time t on the S and R systems. The evolved, entangled, and commuting operators are given from Eq. (44) by

$$R_x(t) = [(c^2 - s^2)R_x + 4csS_xR_y], \tag{45}$$

and

$$S_z(t) = [(c^2 - s^2)S_z - 4csS_yR_z]. \tag{46}$$

Assume that the R-system is aligned along the $+y$ direction in the state $|r> = |+1/2>_y^R$, and note that the expectations $<r|R_z|r>$ and $<r|R_x|r>$ vanish, to obtain the projection of Eqs. (45) and (46) to S system components,

$$<r|R_x(t)|r> = 2scS_x, \tag{47}$$

and

$$<r|S_z(t)|r> = (c^2 - s^2)S_z. \tag{48}$$

The operators $R_x(t)$ and $S_z(t)$ provide an entangled simultaneous measurement of S_x and S_z. The above projections are typically expressed as a partial trace over the meter Hilbert space of the product of the meter density operator $\rho_M = |r><r|$ and the relevant extended-space operator [8]. For example, Eqs. (47) and (48) are expressed as

$$S_x = \frac{1}{2sc} Trace_R[R_x(t)\rho_M], \tag{49}$$

and

$$S_z = \frac{1}{(c^2 - s^2)} Trace_R[S_z(t)\rho_M]. \tag{50}$$

Mathematically, the (S, R)-Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_R$ is the Naimark extension of the S-Hilbert space $\mathcal{H}_S$ [5]-[8]. Among the proposals in this paper is that the Naimark extension is a logically consistent mechanism for joint quantum measurements of non-commuting operators.

C.2 Extended Subspace Lattice

The essential property of an extended quantum system is seen in the lattice of spin-1/2 measurements, which is defined in Refs. [21] and [28]. Consider a spin-1/2 system S and an apparatus that measures spin in the x or z directions with the result up u or down d. As discussed in the previous section, a spin-1/2 meter R is entangled with S such that the R-vacuum expectation value directly reveals the corresponding spin operator of S. Denote by $\alpha\beta\gamma$, with $\alpha \in \{r, s\}$, $\beta \in \{x, z\}$, and $\gamma \in \{u, d\}$, the projected subspace for the measurement on the system α of spin value γ along direction β. For example, sxu defines the subspace generated by the S_x eigenvector with eigenvalue u in system S. A subspace lattice of these outcomes on the system S is shown in Fig. 1, where $\vee(\wedge)$ represents *or* (*and*) connectives on the measurement outcomes [21]. The bold lines in Fig. 1 correspond to measurement outcomes, and the connecting dotted lines denote inclusion into a higher dimensional subspace. The horizontal/vertical and 45°-rotated coordinate systems correspond to S_x and S_z measurements (either u or d), respectively.

A defining feature of quantum lattices is a failure of the distributive property of meet (m) over join (j), which is seen in Fig. 1 by the different outcomes in

$$sxu\, m\, (szu\, j\, szd) = sxu, \tag{51}$$

and

$$(sxu\, m\, szu)\, j\, (sxu\, m\, szd) = \{s\}\, j\, \{s\} = \{s\}, \tag{52}$$

where m and j denote *meet* and *join* operations on the lattice, and the notation $\{\alpha\} = (\alpha xu \wedge \alpha xd) = (\alpha zu \wedge \alpha zd)$ and $[\alpha] = (\alpha xu \vee \alpha xd) = (\alpha zu \vee \alpha zd)$ is used. The failure of the distributive property is due to the fact that the observables S_x and S_z are incompatible; suggesting that the distributive property will hold in an extended lattice with measurements on commuting operators.

The extended Naimark subspace lattice, which combines the S and R Hilbert spaces, is shown in Fig. 2. The vertically-placed coordinates in each row correspond to S (top) and R (bottom) Hilbert spaces that have been entangled to provide simultaneous measurement of system S components. A mechanism for the entangled

Figure 1: Quantum subspace lattice for system S spin measurements. Bold lines are the measurement outcome subspaces. Connecting lines indicate inclusion of lattice subspaces. *Figure attached in file Fig1.gif.*

simultaneous measurement, with R in a special initial state as a meter, was discussed in Section C.1. Consider a section of the extended lattice referring to the simultaneous measurement of the S system spin along the x and z directions. The relevant lattice operations

$$\begin{aligned}(sxu \wedge [r])\, m\, [(szu \wedge [r])\, j\, (rzd \wedge [s])] &= (sxu \wedge [r])\, m\, ([r] \wedge [s]) \\ &= sxu \wedge [r], \end{aligned} \tag{53}$$

and

$$\begin{aligned}&[(sxu \wedge [r])\, m\, (szu \wedge [r])]\, j\, [(sxu \wedge [r])\, m\, (rzd \wedge [s])] \;= \\ &(\{s\} \wedge [r])\, j\, (sxu \wedge rzd) = sxu \wedge [r],\end{aligned} \tag{54}$$

are distributive. All other complex statements mixing S_x on S and S_z on R are distributive in the lattice. For example, another distributive expression is given by

$$\begin{aligned}&(sxu \wedge [r])\, m\, [(\{s\} \wedge rzu)\, j\, (\{s\} \wedge rzd)] = \\ &[(sxu \wedge [r])\, m\, (\{s\} \wedge rzu)]\;\, j\;\, [(sxu \wedge [r])\, m\, (\{s\} \wedge rzd)] \;= \\ &(\{s\} \wedge [r]).\end{aligned} \tag{55}$$

In this appendix a distributive quantum theory is defined using special measurement-dependent statements in cases of incompatible observables. The appropriate measurements involve meters in special-purpose initial states that are entangled with the system. In the body of the paper it was suggested that such measurements are fundamental in the definition of relativistic particle observables. The special meter initial state for particles is the vacuum, the Naimark extension is the repeated fermionic generations, and entanglement is observed in the Cabibbo-Kobayashi-Maskawa matrix and spontaneous breaking of the electro-weak gauge symmetry.

Figure 2: Naimark-extended quantum lattice for entangled simultaneous spin-1/2 measurement with two systems S and R. Subspace for system S outcome is above meter R subspace. Note that $\wedge d$ and $d\wedge$ represent the replacement of u by d in the outcome on the coordinate systems immediately to the left. *Figure attached in file Fig2.gif.*

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